On the derivation of the equations of motion in theories of gravity.

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Abstract

General Relativity is unique, among the class of field theories, in the treatment of the equations of motion. The equations of motion of massive particles are completely determined by the field equation. Einstein's field equations, as well as most field equations in gravity theory, have a specific analytical form: They are linear in the second order derivatives and quadratic in the first order, with coefficients that depend on the variables. We utilize this particular form and propose for the N-body problem of the equations that are Lorentz invariant a novel algorithm for the derivation of the equations of motion from the field equations. It is:

1. Compute a static, spherically symmetric solution of the field equation. It will be singular at the origin. This will be taken to be the field generated by a single particle.
2. Move the solution on a trajectory $\psi(t)$ and apply the instantaneous Lorentz transformation based on instantaneous velocity $\dot{\psi}(t)$.
3. Take, as first approximation, the field generated by N particles to be the superposition of the fields generated by the single particles.
4. Compute the leading part of the equation. Hopefully, only terms that involves $\ddot{\psi}(t)$ will be dominant. This is the "inertial" part.
5. Compute by the quadratic part of the equation. This is the agent of the "force".
6. Equate for each singularity, the highest order terms of the singularities that came from the linear part and the quadratic parts, respectively. This is an equation between the inertial part and the force.

The algorithm was applied to Einstein equations. The approximate evolution of scalar curvature lends, in turn, to an invariant scalar equation. The algorithm for it did produce Newton's law of gravitation. This is, also, the starting point for the embedding the trajectories in a common field.

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1 Introduction

General Relativity (GR) is unique among the class of field theories in the treatment of the equations of motion. The equations of motion of massive particles are completely determined by the field equation. By comparison, classical electrodynamics postulates, in addition to the Maxwell field equations, the force (Coulomb or Lorentz) as well the form of the inertial term.
The derivation of the equation of motion for massive particles from the GR field equations was obtained in the pioneering work of Einstein, Infeld and Hoffman [1] of 1939. The particles were treated as singular points of the metric field. The first order approximation of the field equations resulted in the verification of Newton law of attraction between the singular points. In subsequent papers [3],[4] the higher order approximations to the geodesic line equation (the equations of motion) was calculated. The successive approximations were all derived without additional hypothesis (i.e. the Geodesic Postulate). The method used in the cited papers is known as the *EIH-procedure*. The derivation is rather formal. A similar approach was adopted by Fock [2]. He, however, studied Einstein field equations with the RHS-terms - the tensor of energy-momentum of matter. Sternberg [8] proved the geodesic postulate in a general form for a system that incorporates also Yang-Mills fields. He refers to the establishment of the postulate as to the EIH condition. In his work there is no explicit computation of the trajectories for the N-body problem. The length and complexity of EIH analysis undoubtedly discouraged further research this area.
The derivability of the equations of motion is attributed to two factors:

(i) The GR field equations are non-linear (as opposed to the other classical field equations). As a result, the linear combination of 1-point static solutions does not provide a new static solution.

(ii) The system of the GR field equations is subject to additional identities (Bianchi) which reduce the number of independent equations. The conservation laws (which rest upon these identities) play a crucial role in the original EIH-procedure.

Thus two natural questions arise:

(i) What other non-linear field equations have the property of derivability of the equations of motion?

(ii) Is there a way to embed the moving singularities in a field satisfying the prescribed field equation.

In this paper we deal with the first question and touch upon the second one. First we define a suitable class of the field equations. These are linear in the second order derivatives and quadratic in the first order derivatives both with coefficients that depends only on the field variables. It is also required that the equations are Lorentz invariant. It will be demonstrated in the sequel that the linear part is *the agent of inertia* and the quadratic part is *the agent of the force*. Let the field equation have a static spherical symmetric solution (exact or approximate) which is singular at a point. Accordingly, by the Einstein description, this is the field of one particle. The field of a particle moving with constant velocity is taken to be the Lorentz transform of the static field. If it moves on a curved trajectory $\boldsymbol{\psi}(t)$ then the Lorentz transformation with the instantaneous velocity $\dot{\boldsymbol{\psi}}$ is taken. The field of N particles is taken to be, approximately, the superposition of the fields of the single particles. When this is inserted to the field equation two singular expressions emerge: one from the linear part and the other one from the quadratic part. By equating the highest order terms of these singularities the strength of the singularity of the solution is reduced. It turns out that this balance equation is Newton's law of attraction.

2 The general form of field equations and of their solutions.

We will require the following for the field equation:

(i) PDE of the second order,

(ii) Linear in the second order derivatives with coefficients that depend only on the field variables,

(iii) Quadratic in the first order derivatives.

The last condition can be motivated by the following dimensional argument.[1] Let the field variable Φ will be dimensionless. Thus, the second order derivatives $\Phi_{,\mu\nu}$ have the dimension of $[l^{-2}]$. The only polynomial of the first derivatives that has the same dimension is quadratic in the first order derivatives: $\Phi_{,\mu}\Phi_{,\nu}$. Thus, in order to have a field equation with dimensionless constants we have to consider a PDE, which is linear in the second order derivatives and quadratic in the first order derivatives.
As a consequence we will restrict ourselves to field equations of the form:

$$a\Phi_{,\mu\nu} - b\Phi_{,\mu}\Phi_{,\nu} = 0, \tag{2.1}$$

where the coefficients a and b are the dimensionless functions of the field variable (or constants)

$$a = a(\Phi), \qquad b = b(\Phi).$$

The covariance condition as well as the existence of an action functional provide certain restrictions on the coefficient $a(\Phi)$ and $b(\Phi)$. Note that the form (2.1) is still symbolic, since the field Φ as well as the coefficient functions a and b can possess the interior or coordinate indices and these indices should be contracted in a proper manner.
Note also that the Einstein equation does have the form (2.1).
Let us turn now to the properties of the possible physical relevant solutions of the field equation (2.1). We will apply *the Einstein model of a particle*: it is a smooth solution of the field equation which tends to a constant at infinity and singular at a point. In this model the field pertaining to one particle has only one singularity. In the proper coordinates, the field around the singularity is spherically symmetric and tends to infinity at the singular point. The field pertaining to N particles is approximated by the superposition of rigid motions of of the fields pertaining to all the particles. The trajectory of a singularity, which is taken to be the trajectory of the correspondent particle, is affected by the presence of the fields pertaining to the other particles. This means that the field equation should be non-linear (for a linear equation the fields can not affect each other).
Let us also require that a static spherical symmetric solution of (2.1) has a long distance expansion

$$\Phi = \Phi_0 + \frac{A_1}{r} + \frac{A_2}{r^2} + \cdots + \frac{A_n}{r^n} + \cdots \,, \tag{2.2}$$

where $\{A_n, n = 1, 2, ...\}$ are constants.
It is enough to require the convergence of this expansion for distances greater of some characteristic length of the system. The dimension of A_n is $[l^n]$. A second order equation usually produces two independent constants of integration. Thus the set of constants can be normalized:

$$A_1 = m, \qquad A_n = C_n m^n, \quad n = 2, 3, ..., \tag{2.3}$$

where the dimension of the constant m is equal to $[l]$ and the constants C_i are dimensionless.
Einstein derives the equation of motion by considering the integral relations which are implied by the conservation laws. We propose a different procedure to be exhibited in the sequel.

[1] It was communicated to us by F. W. Hehl.

3 A Qualitative Description of the Algorithm

In this section we describe a novel algorithm for deriving the equations of motion from the field equation. We consider the general field equation (2.1) without specification of the tensorial nature of the field variable Φ. We do require, however, that (2.1) is Lorentz invariant.

Let the field equation (2.1) have a static spherical symmetric solution $\Phi(\mathbf{r}-\mathbf{r}_0)$ with a singularity located at $\mathbf{r}=\mathbf{r}_0$. Denote the Lorentz transformation based on the velocity v by L_v. Consequently, if $\Phi = \Phi(\mathbf{r}-\mathbf{r}_0)$ is a time independent solution of (2.1) then $L_v\Phi$ is also a solution of the same equation for an arbitrary Lorentz transformation L_v. (If Φ is multi-component like a tensor then L_v involves not only a coordinate change but, also, a transformation of components of Φ.) The solution $L_v\Phi$ describes the field of a pointwise singularity moving with a constant velocity v on the trajectory $\boldsymbol{\psi} = \mathbf{v}t$. Let us try to construct a generalization of Lorentz transformation where the origin moves on a general trajectory $\boldsymbol{\psi} = \boldsymbol{\psi}(t)$. Denote such a transformation by N_ψ. The choice

$$N_\psi = L_{\dot\psi}$$

is a plausible candidate. $N_\psi\Phi$ is a rigid motion of the field.

Now substitute $N_\psi\Phi$ in (2.1). If $\dot\psi = const$ then $N_\psi\Phi$ is also a solution of (2.1). If not, then the linear part produces extra terms. These extra terms come from two sources:

- **(i)** *The derivatives of the Lorentz root* $\sqrt{1-\dot\psi^2}$. In our consideration the velocity of the particle $\dot\psi$ as well as its time and spatial derivatives are assumed to be small. It follows that the derivatives of the root are of the form $\mathcal{O}(\ddot\psi\dot\psi)$. Thus they may be discarded.
- **(ii)** *The linear part.* Since Φ is time independent the first derivatives of Φ are only the spatial ones. Thus, the first derivatives of of $L_{\dot\psi}\Phi(x)$ involve spatial derivatives of Φ multiplied by $\dot\psi$. The second derivatives of $L_{\dot\psi}\Phi(x)$ cancel each other the same way as in a Lorentz transformed solution $L_v\Phi(x)$ with one exception: The second derivative $\ddot\psi$ multiplied by spatial first derivatives of Φ do remain. This extra term that come from the linear part is the *agent of inertia*.
- **(iii)** *The quadratic part.* It involves only first order derivatives. Consequently the fact that $\dot\psi$ is variable does not affect it's form.

Construct now a solution that describes the field of N particles, i.e. a field with N singular points. It can be approximated by a superposition of 1-singular solutions moving on arbitrary trajectories ${}^{(j)}\psi = {}^{(j)}\psi(t)$

$$\Phi = \sum_j L_{{}^{(j)}\dot\psi}{}^{(j)}\Phi(x).$$

Substituting this approximate solution in (2.1) we obtain, in the linear part, only the second derivatives ${}^{(j)}\ddot\psi$ multiplied by the spatial first derivatives of Φ.

Consider the quadratic part. It is composed of the first order derivatives of ${}^{(j)}\Phi(x)$ multiplied by the first order derivatives of $L_{{}^{(k)}\Phi}{}^{(k)}\Phi(x)$. If $j = k$, since ${}^{(k)}\Phi(x)$ is a solution of (2.1), these products are cancelled by the linear part operating on ${}^{(k)}\Phi(x)$. If $j \neq k$ the products will, hopefully, be an approximation to the interaction between the j-th and the k-th particles.

Near the k-th singularity, for the linear part, only the terms coming from ${}^{(k)}\Phi(x)$ will be dominant. Likewise, for the quadratic part, only the terms involving the derivatives of ${}^{(k)}\Phi(x)$ will be dominant. Equating, near the singularity, of the two terms above (again to the leading order) should, hopefully, result in the Newtonian law of attraction.

Let us summarize the novel algorithm for the derivation of equations of motion.

1 Compute a static, spherically symmetric solution of the field equation. It will be singular at the origin. This will be taken to be the field generated by a single particle.

2 Move the solution on a trajectory $\boldsymbol{\psi}(t)$ and apply the instantaneous Lorentz transformation based on $\dot{\boldsymbol{\psi}}(t)$.

3 Take the field generated by n particles to be the superposition of the fields generated by the single particles.

4 Compute the leading part of the equation. Hopefully, only terms that involves $\ddot{\boldsymbol{\psi}}$ will be dominant.

5 Compute the "force" between the particles by the quadratic part of the equation.

6 Equate for each singularity, the highest order terms of the singularities that came from the linear part and the quadratic parts, respectively. This is an equation between the inertial part and the force.

4 Einsteinian gravity

Let us apply the procedure described above to the Einsteinian gravity. Our first step is to construct an approximated dynamical N-particle solution from the static 1-particle solution. The Einstein field equation in vacuo is

$$R_{ik} = 0. \tag{4.1}$$

Consider the Schwarzschild solution in isotropic coordinates

$$ds^2 = \Big(\frac{1-m/4r}{1+m/4r}\Big)^2 dt^2 - (1+m/4r)^4(dx^2+dy^2+dz^2). \tag{4.2}$$

We will use the general diagonal metric of the form

$$ds^2 = e^{2f}dt^2 - e^{2g}(dx^2+dy^2+dz^2). \tag{4.3}$$

The static solution (4.2) has the long-distance expansion

$$f = -\frac{m}{2r} + \mathcal{O}\Big(\frac{m}{r}\Big)^3 \tag{4.4}$$

$$g = \frac{m}{2r} - \frac{1}{16}\Big(\frac{m}{r}\Big)^2 + \mathcal{O}\Big(\frac{m}{r}\Big)^3. \tag{4.5}$$

Hence, up to $\mathcal{O}\Big(\frac{m}{r}\Big)^3$, the following relation holds

$$g = -f - \frac{1}{4}f^2. \tag{4.6}$$

We will seek the time depended solution of the field equation (4.1) in the diagonal form (4.3) with the functions $f = f(x^i, t)$ and $g = g(x^i, t)$.
Let us assume that the static relation (4.6) holds for the time dependent metric as well.
Thus, in order to determine the metric up to $\mathcal{O}(\frac{m}{r})^3$, it is enough to find the function f. This way, by

simplifying the assumptions above, the solution of (4.1) is, approximately, dependent on one function f. If this is needed we can look at the scalar covariant equation

$$R = 0. \tag{4.7}$$

Let us first insert the diagonal metric (4.3) in (4.7). The calculations result in

$$R = -6e^{-2f}(\ddot{g} - \dot{f}\dot{g} + 2\dot{g}^2) + 2e^{-2g}\Big(\triangle f + 2\triangle g + (\nabla f)^2 + \nabla f \cdot \nabla g + (\nabla g)^2\Big) \tag{4.8}$$

($\dot{f}$ is used for for the time derivative of f).
Insert the relation (4.6) to this equation. The result is

$$R = 3e^{-2f}\Big((2-f)\ddot{f} - (7+5f+f^2)\dot{f}^2\Big) + e^{2f+1/2f^2}\Big(-2(1+f)\triangle f + (f + \frac{1}{2}f^2)(\nabla f)^2\Big). \tag{4.9}$$

Estimate the terms of this equation. Note, first, that the the term containing $\triangle f$ vanishes for the static as well as the time dependent solution obtained from it by Lorentz type transformations. As for the gradient term, it is estimated by

$$f(\nabla f)^2 \sim \frac{1}{r^2}\mathcal{O}(\frac{m}{r})^3.$$

(Note that the equation does not include the term $\frac{1}{r^2}\mathcal{O}(\frac{m}{r})^2$ which may come from the expression $(\nabla f)^2$.) Thus, up to $\frac{1}{r^2}\mathcal{O}(\frac{m}{r})^3$, the curvature scalar is

$$R = 6\ddot{f} - 2\triangle f. \tag{4.10}$$

Be rescaling the coordinates we obtain the Lorentz invariant equation

$$R = 2\,\Box f = 0. \tag{4.11}$$

Recall that the truly dynamical variable are the metric components determined by the function $\varphi = e^{2f}$. The equation for this function is

$$\Box\varphi = \frac{1}{\varphi}(\nabla\varphi)^2. \tag{4.12}$$

Note that the function $\varphi = 1 + \frac{m}{R}\mathcal{O}(1)$. Hence up to $\frac{1}{r^2}\mathcal{O}(\frac{m}{r})^3$ the equation (4.12) takes the form

$$\Box\varphi = (\nabla\varphi)^2. \tag{4.13}$$

This scalar equation is Lorentz invariant and non-linear. In the next section it will be shown that the algorithm is applicable for this equation.

5 A Non-linear scalar model

Let a flat Minkowskian $4D$-space with a metric $\eta_{ab} = \eta^{ab} = diag(1,-1,-1,-1)$ and a scalar field $f = f(x)$ be given. Consider, first, the wave equation - a linear second order equation for the field f

$$\Box f = *d*df = 0. \tag{5.1}$$

It has a unique 1-singular at the origin spherical symmetric and asymptotically zero solution

$$f = \frac{m}{R}. \tag{5.2}$$

It is natural to interpret this solution as the field of a pointwise body located at the origin. The linearity of the field equation (5.1) yields the existence of N-singular asymptotically zero solution

$$f = \sum_{i=1}^{n} \frac{m_i}{|\boldsymbol{R}_i|}, \qquad \boldsymbol{R}_i = \mathbf{r} - \mathbf{r}_i. \tag{5.3}$$

The solution (5.3) have to be interpreted as a field produced by a static configuration of bodies of masses m_i located at the points $\mathbf{r}_i$. This interpretation is, of course, not physical because it completely ignores the interaction between the particles. One may attempt to apply now an instantaneous Lorentz transformation (with a time dependent velocity) of the static solution in order to produce a solution with interactions. For that, let the singular point move on an arbitrary trajectory $\boldsymbol{\psi}(t)$. Consider the function

$$f = \sum_{i} \frac{m_i}{|\boldsymbol{R}_i|}, \tag{5.4}$$

where

$$\boldsymbol{R}_i = (\mathbf{r} - \mathbf{r}_i) - \alpha_i \dot{\boldsymbol{\psi}}_i (\dot{\boldsymbol{\psi}}_i, (\mathbf{r} - \mathbf{r}_i)) - \beta_i \boldsymbol{\psi}_i, \tag{5.5}$$

and the Lorentz parameters α_i and β_i are functions only of $|\dot{\boldsymbol{\psi}}_i|^2$

$$\beta_i = \frac{1}{\sqrt{1 - |\dot{\boldsymbol{\psi}}_i|^2}}, \qquad \alpha_i = \frac{1}{|\dot{\boldsymbol{\psi}}_i|^2}\Big(1 - \frac{1}{\sqrt{1 - |\dot{\boldsymbol{\psi}}_i|^2}}\Big) \tag{5.6}$$

This is the form of Lorentz transformation for an arbitrary velocity (Cf. Appendix A).
For $\boldsymbol{\psi}_i = \dot{\boldsymbol{\psi}}_i t$, at the point $\boldsymbol{R}_i = 0$

$$\mathbf{r} - \mathbf{r}_i = \dot{\boldsymbol{\psi}}_i \Big(\alpha_i (\dot{\boldsymbol{\psi}}_i, (\mathbf{r} - \mathbf{r}_i)) + \beta_i t\Big). \tag{5.7}$$

So

$$(\dot{\boldsymbol{\psi}}_i, (\mathbf{r} - \mathbf{r}_i)) = |\dot{\boldsymbol{\psi}}_i|^2 \Big(\alpha_i (\dot{\boldsymbol{\psi}}_i, (\mathbf{r} - \mathbf{r}_i)) + \beta_i t\Big). \tag{5.8}$$

Or

$$(\dot{\boldsymbol{\psi}}_i, (\mathbf{r} - \mathbf{r}_i)) = \frac{|\dot{\boldsymbol{\psi}}_i|^2 \beta_i t}{1 - \alpha_i |\dot{\boldsymbol{\psi}}_i|^2}. \tag{5.9}$$

Substituting this relation in (5.7) we obtain the equation motion of the singularity as

$$\mathbf{r} - \mathbf{r}_i = \frac{\beta_i}{1 - \alpha_i |\dot{\boldsymbol{\psi}}_i|^2} \dot{\boldsymbol{\psi}}_i t = \dot{\boldsymbol{\psi}}_i t, \tag{5.10}$$

which is the motion of a free particle.
Calculate approximately the d'Alembertian of the function (5.4), in particular, omit the time derivatives of α_i and β_i to obtain (Cf. Appendix B)

$$\Box f = \sum_{i=1}^{n} \frac{m_i \beta_i}{R_i^3} (\ddot{\boldsymbol{\psi}}_i, \boldsymbol{R}_i) + \mathcal{O}(\ddot{\boldsymbol{\psi}} \dot{\boldsymbol{\psi}}). \tag{5.11}$$

Thus, for the field equation (5.1) f is a solution if and only if $\ddot{\boldsymbol{\psi}}_i = 0$.
Consequently, the linear equation describes a free inertial motion of an arbitrary system of singularities. Consider now a nonlinear covariant field equation

$$d * df = k df \wedge * df, \tag{5.12}$$

or, in coordinate form

$$\Box f = k\eta^{ab} f_{,a} f_{b}. \tag{5.13}$$

It is easy to see that the the field equation (5.13) can be transformed to the field equation (5.1) by the redefinition of the scalar field

$$\varphi = e^{-kf}. \tag{5.14}$$

If the field f satisfies the equation (5.13) the new field φ satisfies (5.1). The transformation (5.14) can be used in order to derive a close form solution to the nonlinear equation (5.13).

$$f = -\frac{1}{k} \ln\Big(1 + k\frac{m}{|\boldsymbol{R}|}\Big). \tag{5.15}$$

In order to have a singularity only at the origin we have to require the constants k and m to be positive. The solution (5.15) is singular at the origin and vanishes at the infinity. It's long-distance expansion begins with m/r. In the limit case $k \to 0$ the solution (5.15) approaches the Newtonian potential $f \to \frac{m}{r}$. Because of the transformation (5.14) the equation (5.13) has also a static N-singular points solution

$$f = -\frac{1}{k} \ln\Big(1 + k\sum_{i=1}^{N} \frac{{}^{(i)}m}{{}^{(i)}R}\Big), \tag{5.16}$$

where ${}^{(i)}\boldsymbol{R} = \mathbf{r} - \mathbf{r}_i$ and $\mathbf{r}_i$ are the radius vectors of the N fixed points.
The solution (5.16) is static. Let us look for an approximate dynamic one. In order to construct an approximate solution take the superposition of the fields

$$f = -\frac{1}{k} \sum_{i=1}^{N} \ln\Big(1 + k\frac{{}^{(i)}m}{{}^{(i)}R}\Big). \tag{5.17}$$

Let us apply the Lorentz transformation with variable velocities. Now the vector $\boldsymbol{R}$ c.f. (5.5) is time dependent. Calculate the linear part of the equation (Cf. Appendix C) to obtain

$$\Box f = \sum_{i=1}^{n} \Big(\frac{{}^{(i)}m {}^{(i)}\beta}{{}^{(i)}R^3} \cdot \frac{({}^{(i)}\boldsymbol{R}, {}^{(i)}\ddot{\boldsymbol{\psi}})}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} - \frac{{}^{(i)}m^2 k}{{}^{(i)}R^4} \frac{1}{(1 - k\frac{{}^{(i)}m}{{}^{(i)}R})^2} \Big). \tag{5.18}$$

By Appendix D the nonlinear part is:

$$\begin{aligned} k\eta^{ab} f_{,a} f_{,b} \quad &= \quad -k\sum_{i} \frac{{}^{(i)}m^2 k}{{}^{(i)}R^4} \frac{1}{(1 - k\frac{{}^{(i)}m}{{}^{(i)}R})^2} - \sum_{i\neq j} \frac{\frac{{}^{(i)}m}{{}^{(i)}R^3}}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} \cdot \frac{\frac{{}^{(j)}m}{{}^{(j)}R^3}}{1 - k\frac{{}^{(j)}m}{{}^{(j)}R}} \Big[({}^{(i)}\boldsymbol{R}, {}^{(j)}\boldsymbol{R}) + \\ &\quad ({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(j)}\boldsymbol{R})({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R}) \Big({}^{(i)}\beta {}^{(j)}\beta + {}^{(i)}\alpha + {}^{(j)}\alpha - {}^{(i)}\alpha {}^{(j)}\alpha ({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(j)}\dot{\boldsymbol{\psi}}) \Big) \Big]. \end{aligned}$$

The field equation takes the form

$$\begin{aligned} &\sum_{i=1}^{n} \frac{{}^{(i)}m {}^{(i)}\beta}{{}^{(i)}R^3} \cdot \frac{({}^{(i)}\boldsymbol{R}, {}^{(i)}\ddot{\boldsymbol{\psi}})}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} = -k\sum_{i\neq j} \frac{\frac{{}^{(i)}m}{{}^{(i)}R^3}}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} \cdot \frac{\frac{{}^{(j)}m}{{}^{(j)}R^3}}{1 - k\frac{{}^{(j)}m}{{}^{(j)}R}} \Big[({}^{(i)}\boldsymbol{R}, {}^{(j)}\boldsymbol{R}) + \\ &({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(j)}\boldsymbol{R})({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R}) \Big({}^{(i)}\beta {}^{(j)}\beta + {}^{(i)}\alpha + {}^{(j)}\alpha - {}^{(i)}\alpha {}^{(j)}\alpha ({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(j)}\dot{\boldsymbol{\psi}}) \Big) \Big] + \mathcal{O}(\ddot{\psi}\dot{\psi}). \end{aligned} \tag{5.19}$$

For the approximation of slow motions it is

$$\sum_{i=1}^{n} \frac{{}^{(i)}m}{{}^{(i)}R^3} \cdot \frac{({}^{(i)}\boldsymbol{R}, {}^{(i)}\ddot{\boldsymbol{\psi}})}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} = -k\sum_{i\neq j} \frac{\frac{{}^{(i)}m}{{}^{(i)}R^3}}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} \cdot \frac{\frac{{}^{(j)}m}{{}^{(j)}R^3}}{1 - k\frac{{}^{(j)}m}{{}^{(j)}R}} ({}^{(i)}\boldsymbol{R}, {}^{(j)}\boldsymbol{R}) + \mathcal{O}(\ddot{\psi}\dot{\psi}) + \frac{1}{R^2}\mathcal{O}(\frac{m}{R})^2. \tag{5.20}$$

The two sides of this equation are functions of the variable x. Consider the p-th singularity. Take an arbitrary point x close to this singularity. It follows that

$$ {}^{(p)}\boldsymbol{R} \to 0 \quad \text{and} \quad {}^{(i)}\boldsymbol{R} \to \bar{R}_{ip} \quad \text{for} \quad i \neq p, \tag{5.21} $$

where $\bar{R}_{ip}$ is a vector from the point i to the point p.
In the LHS of the equation (5.20) there is one singular term

$$ \frac{{}^{(p)}m}{{}^{(p)}R^3} \frac{({}^{(p)}\boldsymbol{R}, {}^{(p)}\ddot{\boldsymbol{\psi}})}{1 - k\frac{{}^{(p)}m}{{}^{(p)}R}}. \tag{5.22} $$

The singular term in the RHS of (5.20) is

$$ -k\frac{\frac{{}^{(p)}m}{{}^{(p)}R^3}}{1 - k\frac{{}^{(p)}m}{{}^{(p)}R}} \sum_{j\neq p} \frac{\frac{{}^{(j)}m}{{}^{(j)}R^3}}{1 - k\frac{{}^{(j)}m}{{}^{(j)}R}} ({}^{(p)}\boldsymbol{R}, {}^{(j)}\boldsymbol{R}). \tag{5.23} $$

The terms (5.22) and (5.23) are $\mathcal{O}(R^{-2})$ near the singularity. When these are inserted to the RHS and LHS of (5.20), respectively, the remainder will be $\mathcal{O}(R^{-1})$ if only if:

$$ ({}^{(p)}\boldsymbol{R}, {}^{(p)}\ddot{\boldsymbol{\psi}}) = -k \sum_{j\neq p} \frac{\frac{{}^{(j)}m}{{}^{(j)}R^3}}{1 - k\frac{{}^{(j)}m}{{}^{(j)}R}} ({}^{(p)}\boldsymbol{R}, {}^{(j)}\boldsymbol{R}) \tag{5.24} $$

This is the only way to diminish the strength of the singularity at (5.20).
Take into account that the point x is still arbitrary. Hence (5.24) is valid only if

$$ {}^{(p)}\ddot{\boldsymbol{\psi}} = -k \sum_{j\neq p} \frac{\frac{{}^{(j)}m}{{}^{(j)}R^3}}{1 - k\frac{{}^{(j)}m}{{}^{(j)}R}} {}^{(j)}\boldsymbol{R} \tag{5.25} $$

For the limiting values in (5.21)

$$ {}^{(p)}\ddot{\boldsymbol{\psi}} = -k \sum_{j\neq p} \frac{{}^{(j)}m\boldsymbol{R}_{jp}}{R_{jp}^3} \frac{1}{1 - k\frac{{}^{(j)}m}{R_{jp}}} \tag{5.26} $$

The second fraction differs from 1, significantly, only for small distances comparable to the Schwarzschild radius $r = km$. Thus, it can be neglected.
It follows that

$$ {}^{(p)}\ddot{\boldsymbol{\psi}} = -k \sum_{j\neq p} \frac{{}^{(j)}m\boldsymbol{R}_{jp}}{R_{jp}^3} \tag{5.27} $$

For a system of two singular points

$$ {}^{(1)}\ddot{\boldsymbol{\psi}} = -k\frac{{}^{(2)}m\boldsymbol{R}_{21}}{R_{21}^3} \tag{5.28} $$

For $k < 0$ (5.27) and (5.28) result in attraction between the particles. The absolute value of k is unimportant, since it amounts to the rescaling of the mass.
This way Newton's law is obtained.

6 The Embedding problem

The trajectories obtained are approximate. At this point, two avenues are open. The first one, which is adopted by EIH is to get higher order approximations to the trajectories. This procedure is also used in the PPN approach. By these methods, the successive approximations become highly singular near the particle trajectories.
The second avenue is to embed the singularities in a field satisfying the field equations. For that purpose, the successive approximations should add regular terms (and, possible, low order singular terms) near the trajectories. In this article we just touch upon the method to be developed. Consider a first order solution i.e. the trajectories are taken to be fixed and the calculations is done only for the first order. For the scalar model the desired field will be a solution φ of (5.13),

$$\varphi = e^{-f+g}, \tag{6.1}$$

where f is defined by (5.17) and g is regular at the vicinity of all the singularities ("multi Green function"). Let us present a formal argument that shows that the construction is possible. Take, approximately $f = \mathcal{O}(\frac{m}{r})$It follows from the computations in the previous section that

$$(\Box f - \eta^{ab} f_{,a} f_{,b}) = mF \tag{6.2}$$

(5.13) is satisfied when

$$\Box g - 2\eta^{ab} f_{,a} g_{,b} + \eta^{ab} g_{,a} g_{,b} = -mF. \tag{6.3}$$

Newton's law holds, thus $F = \mathcal{O}(\frac{1}{r})$. Consider for F two examples:

$$F = \frac{m}{r} + \text{regular terms}$$

or

$$F = \frac{m < r, a >^2}{r^3} + \text{regular terms}$$

In the first case, take $g^{(1)} = -\frac{1}{2} c r^2$, in the second case (a is a constant vector) take $g^{(1)} = -\frac{1}{2} c < r, a > r$. In both cases the remainder in the left hand side of (5.13) will be regular.

A Lorentz transformations

The Lorentz transformations are usually written in a very special case when the axes of two reference systems are parallel and one of the systems moves relative to the other with a velocity parallel to the x-axis [2]

$$x = \alpha(\tilde{x} + v\tilde{t}), \quad y = \tilde{y}, \quad z = \tilde{z}, \quad t = \alpha(\tilde{t} + v\tilde{x}), \tag{A.1}$$

with the Lorentz parameter

$$\alpha = (1 - v^2)^{-\frac{1}{2}}. \tag{A.2}$$

It is well known that the Lorentz transformations are non-commutative. Consequently, a general transformation with an arbitrary directed vector of velocity can not be generated as a successive application

[2] Recall that we use the system of units with $c = 1$

of three orthogonal transformations (relative to the axes).
In the sequel, a formula for Lorentz transformation for general velocity vector is exhibited. It is hard to believe that such formula does not exists in the literature. The authors, however, could not find a reference for it.
Consider a reference system moving with an arbitrary directed velocity $\mathbf{v}$ with the axes parallel to the corresponding axes of a rest reference system. Consider a vector $\mathbf{r}$ to an arbitrary point in space. The projection of the vector $\mathbf{r}$ on the direction $\mathbf{v}$ will be

$$P_{\mathbf{v}}\mathbf{r} = \frac{\mathbf{v}(\mathbf{v},\mathbf{r})}{v^2}. \tag{A.3}$$

Expand the vector $\mathbf{r}$ as

$$\mathbf{r} = P_{\mathbf{v}}\mathbf{r} + N_{\mathbf{v}}\mathbf{r}. \tag{A.4}$$

For a Lorentz transformation

$$P_{\mathbf{v}}\tilde{\mathbf{r}} = \alpha(P_{\mathbf{v}}\mathbf{r} + \mathbf{v}t), \tag{A.5}$$
$$N_{\mathbf{v}}\tilde{\mathbf{r}} = N_{\mathbf{v}}\mathbf{r}. \tag{A.6}$$

Consequently, the transform of the spatial coordinates is

$$\tilde{\mathbf{r}} = \alpha(P_{\mathbf{v}}\mathbf{r} + \mathbf{v}t) + N_{\mathbf{v}} \tag{A.7}$$

or, explicitly

$$\tilde{\mathbf{r}} = \mathbf{r} + \mathbf{v}\Big(\alpha t - \frac{(1-\alpha)}{v^2}(\mathbf{r},\mathbf{v})\Big). \tag{A.8}$$

The change of the time coordinate is also governed only by the tangential part of the vector $\mathbf{r}$

$$\tilde{t} = \alpha(t + |P_{\mathbf{v}}\mathbf{r}|v) = \alpha\Big(t + (P_{\mathbf{v}}\mathbf{r},\mathbf{v})\Big) \tag{A.9}$$

or explicitly

$$\tilde{t} = \alpha\Big(t + (\mathbf{r},\mathbf{v})\Big). \tag{A.10}$$

In the special case of a motion parallel to the axis x the relations (A.8), (A.10) reduce to the ordinary form of Lorentz transformation (A.1).
Therefore an arbitrary Lorentz transformation takes the form

$$\begin{cases} \tilde{t} = \alpha\Big(t + (\mathbf{r},\mathbf{v})\Big) \\ \tilde{\mathbf{r}} = \mathbf{r} + \mathbf{v}\Big(\alpha t - \frac{(1-\alpha)}{v^2}(\mathbf{r},\mathbf{v})\Big) \end{cases} \tag{A.11}$$

Taking the derivative of (A.11) the law of transformations of the differentials is obtained

$$\begin{cases} d\tilde{t} = \alpha\Big(dt - v_j dx^j\Big) \\ d\tilde{x}^i = dx^i + v^i\Big(\alpha dt + \frac{(1-\alpha)}{v^2}v_j dx^j\Big) \end{cases} \tag{A.12}$$

This can be represented in a matrix form by

$$d\tilde{x} = A dx \qquad <===> \qquad d\tilde{x}^\mu = A^\mu{}_\nu dx^\nu \tag{A.13}$$

with

$$A = \begin{pmatrix} \alpha & \alpha v^1 & \alpha v^2 & \alpha v^3 \\ \alpha v^1 & 1+(1-\alpha)\frac{v_1^2}{v^2} & (1-\alpha)\frac{v_1 v_2}{v^2} & (1-\alpha)\frac{v_1 v_3}{v^2} \\ \alpha v^2 & (1-\alpha)\frac{v_1 v_2}{v^2} & 1+(1-\alpha)\frac{v_2^2}{v^2} & (1-\alpha)\frac{v_1 v_3}{v^2} \\ \alpha v^3 & (1-\alpha)\frac{v_1 v_3}{v^2} & (1-\alpha)\frac{v_2 v_3}{v^2} & 1+(1-\alpha)\frac{v_3^2}{v^2} \end{pmatrix} \tag{A.14}$$

For our purposes it is enough to have the second order ($\frac{1}{c^2}$) approximation of the transforms (A.8), (A.10). Use

$$\alpha = \frac{1}{\sqrt{1-v^2}} = 1 + \frac{1}{2}v^2 + O(v^4) \tag{A.15}$$

to obtain

$$\begin{cases} \tilde{\mathbf{r}} = \mathbf{r} + \mathbf{v}t + \mathbf{v}(\mathbf{r},\mathbf{v}) \\ \tilde{t} = (1-\frac{1}{2}v^2)t + (\mathbf{r},\mathbf{v}). \end{cases} \tag{A.16}$$

Consequently the approximation of the transformation law for the differentials is

$$\begin{cases} d\tilde{t} = (1+\frac{1}{2}v^2)dt - v_j dx^j \\ d\tilde{x}^i = dx^i + v^i\Big(dt - \frac{1}{2}v_j dx^j\Big). \end{cases} \tag{A.17}$$

Thus the matrix of transformation is

$$A = \begin{pmatrix} 1+\frac{1}{2}v^2 & v^1 & v^2 & v^3 \\ v^1 & 1-\frac{1}{2}v_1^2 & -\frac{1}{2}v_1 v_2 & \frac{1}{2}v_1 v_3 \\ v^2 & -\frac{1}{2}v_1 v_2 & 1-\frac{1}{2}v_2^2 & -\frac{1}{2}v_1 v_3 \\ v^3 & -\frac{1}{2}v_1 v_3 & \frac{1}{2}v_2 v_3 & 1-\frac{1}{2}v_3^2. \end{pmatrix}. \tag{A.18}$$

B 1-point ansatz

Calculate the d'Alembertian of the field f of a moving particle f:

$$f = \frac{m}{R}, \tag{B.1}$$

where

$$\boldsymbol{R} = (\mathbf{r}-\mathbf{r}_0) - \alpha\dot{\overline{\psi}}(\dot{\boldsymbol{\psi}},(\mathbf{r}-\mathbf{r}_0)) - \beta\overline{\psi}, \tag{B.2}$$

with $\overline{\psi} = \overline{\psi}(t)$ and α and β are functions of $|\dot{\boldsymbol{\psi}}|^2$:

$$\beta = \frac{1}{\sqrt{1-|\dot{\boldsymbol{\psi}}|^2}}, \qquad \alpha = \frac{1}{|\dot{\boldsymbol{\psi}}|^2}\Big(1 - \frac{1}{\sqrt{1-|\dot{\boldsymbol{\psi}}|^2}}\Big) \tag{B.3}$$

By straightforward calculation

$$\begin{aligned} \boldsymbol{R}_t \quad = \quad & -2\alpha'\Big((\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\psi}})\dot{\boldsymbol{\psi}}(\dot{\boldsymbol{\psi}},\mathbf{r}-\mathbf{r}_0)\Big) - \alpha\Big(\ddot{\boldsymbol{\psi}}(\dot{\boldsymbol{\psi}},\mathbf{r}-\mathbf{r}_0)\Big) \\ & -\alpha\Big(\dot{\boldsymbol{\psi}}(\ddot{\boldsymbol{\psi}},\mathbf{r}-\mathbf{r}_0)\Big) - 2\beta'\Big((\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\psi}})\overline{\psi}\Big) - \beta\dot{\boldsymbol{\psi}} \end{aligned} \tag{B.4}$$

and

$$\begin{aligned}
\boldsymbol{R}_{tt} &= -4\alpha''\Big((\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\psi}})^2(\dot{\boldsymbol{\psi}},(\mathbf{r}-\mathbf{r}_0))\Big) - 2\alpha'\Big(|\ddot{\boldsymbol{\psi}}|^2(\dot{\boldsymbol{\psi}},(\mathbf{r}-\mathbf{r}_0))\Big) \\
&\quad -2\alpha'\Big((\dot{\boldsymbol{\psi}},\dddot{\boldsymbol{\psi}})(\dot{\boldsymbol{\psi}},(\mathbf{r}-\mathbf{r}_0))\Big) - 2\alpha'\Big((\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\psi}})(\ddot{\boldsymbol{\psi}},(\mathbf{r}-\mathbf{r}_0))\Big) \\
&\quad -4\beta''\Big((\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\psi}})^2\boldsymbol{\psi}\Big) - 2\beta'\Big(|\ddot{\boldsymbol{\psi}}|^2\boldsymbol{\psi}\Big) - 2\beta'\Big((\dot{\boldsymbol{\psi}},\dddot{\boldsymbol{\psi}})\boldsymbol{\psi}\Big) - 2\beta'\Big((\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\psi}})\dot{\boldsymbol{\psi}}\Big) \\
&\quad -2\beta'\Big((\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\psi}})\dot{\boldsymbol{\psi}}\Big) - \beta\ddot{\boldsymbol{\psi}}.
\end{aligned} \tag{B.5}$$

Thus

$$\begin{aligned}
f_t &= -\frac{m}{R^3}(\boldsymbol{R}_t,\boldsymbol{R}) = -\frac{m}{R^3}\Bigg(-2\alpha'\Big((\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\psi}})(\dot{\boldsymbol{\psi}},\boldsymbol{R})(\dot{\boldsymbol{\psi}},\mathbf{r}-\mathbf{r}_0)\Big) - \\
&\quad \alpha\Big((\ddot{\boldsymbol{\psi}},\boldsymbol{R})(\dot{\boldsymbol{\psi}},\mathbf{r}-\mathbf{r}_0)\Big) - \alpha\Big((\dot{\boldsymbol{\psi}},R)(\ddot{\boldsymbol{\psi}},\mathbf{r}-\mathbf{r}_0)\Big) \\
&\quad -2\beta'\Big((\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\psi}})(\overline{\psi},\boldsymbol{R})\Big) - \beta\Big((\dot{\boldsymbol{\psi}},R)\Big)\Bigg)
\end{aligned} \tag{B.6}$$

In the first approximation (for velocities that are small with respect to the speed of the light) we can take

$$\boldsymbol{R}_t = -\beta\dot{\boldsymbol{\psi}}, \qquad \boldsymbol{R}_{tt} = -\beta\ddot{\boldsymbol{\psi}}. \tag{B.7}$$

Consequently,

$$f_t = \frac{m}{R^3}\beta(\dot{\boldsymbol{\psi}},R) \tag{B.8}$$

In general, the derivatives of α and β contribute terms that are quadratic in $\dot{\boldsymbol{\psi}}$ and its derivatives. Hence, they can be neglected. Thus, the second time derivative, to the same accuracy, is

$$f_{tt} = \frac{m\beta}{R^3}\Big((\ddot{\boldsymbol{\psi}},\boldsymbol{R}) + (\dot{\boldsymbol{\psi}},\boldsymbol{R}_t)\Big) - 3\frac{m\beta}{R^5}(\dot{\boldsymbol{\psi}},\boldsymbol{R})(\boldsymbol{R}_t,\boldsymbol{R}) \tag{B.9}$$

Substitute

$$\boldsymbol{R}_t = -\beta\dot{\boldsymbol{\psi}} \tag{B.10}$$

to get

$$\begin{aligned}
f_{tt} &= \frac{m\beta}{R^3}\Big((\ddot{\boldsymbol{\psi}},\boldsymbol{R}) - \beta\ (\dot{\boldsymbol{\psi}},\dot{\boldsymbol{\psi}})\Big) \ +\ 3\frac{m\beta^2}{R^5}(\dot{\boldsymbol{\psi}},\boldsymbol{R})^2 \\
&= \frac{m\beta}{R^3}(\ddot{\boldsymbol{\psi}},\boldsymbol{R}) \ +\ \frac{m\beta^2}{R^5}\Big(3(\dot{\boldsymbol{\psi}},\boldsymbol{R})^2 - R^2|\dot{\boldsymbol{\psi}}|^2\Big).
\end{aligned} \tag{B.11}$$

As for the spatial derivatives

$$\boldsymbol{R}_x = \boldsymbol{e}_1 - \alpha\dot{\psi}_1\dot{\boldsymbol{\psi}}, \qquad\qquad \boldsymbol{R}_{xx} = 0 \tag{B.12}$$

$$f_x = -\frac{m}{R^3}(\boldsymbol{R}_x,\boldsymbol{R}) \tag{B.13}$$

$$f_{xx} = -\frac{m}{R^3}(\boldsymbol{R}_x,\boldsymbol{R}_x) \ +\ 3\frac{m}{R^5}(\boldsymbol{R}_x,\boldsymbol{R})^2 \tag{B.14}$$

$$\triangle\ f = 3\frac{m}{R^5}\Big((\boldsymbol{R}_x,\boldsymbol{R})^2 + (\boldsymbol{R}_y,\boldsymbol{R})^2 + (\boldsymbol{R}_z,\boldsymbol{R})^2\Big) - \frac{m}{R^3}\Big(R_x^2 + R_y^2 + R_z^2\Big) \tag{B.15}$$

Since

$$(\boldsymbol{R}_x,\boldsymbol{R}) = R_1 - \alpha\dot{\psi}_1(\dot{\boldsymbol{\psi}}, R) \tag{B.16}$$

we get

$$(\boldsymbol{R}_x,\boldsymbol{R})^2 + (\boldsymbol{R}_y,\boldsymbol{R})^2 + (\boldsymbol{R}_z,\boldsymbol{R})^2 = R^2 - 2\alpha(\dot{\boldsymbol{\psi}}, R)^2 + \alpha^2|\dot{\boldsymbol{\psi}}|^2(\dot{\boldsymbol{\psi}}, R)^2 \tag{B.17}$$

and

$$R_x^2 + R_y^2 + R_z^2 = 3 - 2\alpha|\dot{\boldsymbol{\psi}}|^2 + \alpha^2|\dot{\boldsymbol{\psi}}|^4 \tag{B.18}$$

Thus the Laplacian is

$$\triangle f = \frac{m}{R^5}(\alpha^2|\dot{\boldsymbol{\psi}}|^2 - 2\alpha)\Big(3(\dot{\boldsymbol{\psi}}, R)^2 - |\dot{\boldsymbol{\psi}}|^2 R^2\Big) \tag{B.19}$$

And, for the d'Alembertian

$$\Box f = \frac{m\beta}{R^3}(\ddot{\boldsymbol{\psi}},\boldsymbol{R}) + \frac{m}{R^5}(\beta^2 + 2\alpha - \alpha^2|\dot{\boldsymbol{\psi}}|^2)\Big(3(\dot{\boldsymbol{\psi}},\boldsymbol{R})^2 - R^2|\dot{\boldsymbol{\psi}}|^2\Big) \tag{B.20}$$

Using the expressions for the functions α, β we obtain, approximately,

$$\Box f = \frac{m\beta}{R^3}(\ddot{\boldsymbol{\psi}},\boldsymbol{R}) \tag{B.21}$$

C The leading part

From the calculations above

$$\boldsymbol{R}_t = -\beta\dot{\boldsymbol{\psi}}, \qquad \boldsymbol{R}_{tt} = -\beta\ddot{\boldsymbol{\psi}}. \tag{C.1}$$

Thus

$$f_t = -\frac{m}{R^3}\frac{(\boldsymbol{R},\boldsymbol{R}_t)}{1 - k\frac{m}{R}} = \frac{m\beta}{R^3}\frac{(\boldsymbol{R},\dot{\boldsymbol{\psi}})}{1 - k\frac{m}{R}}. \tag{C.2}$$

And

$$f_{tt} = \frac{m\beta}{R^5}\cdot\frac{3\beta(\boldsymbol{R},\dot{\boldsymbol{\psi}})^2 + R^2(\boldsymbol{R},\ddot{\boldsymbol{\psi}}) - \beta R^2|\dot{\boldsymbol{\psi}}|^2}{1 - k\frac{m}{R}} + \frac{m^2\beta^2 k}{R^6}\cdot\frac{(\boldsymbol{R},\dot{\boldsymbol{\psi}})^2}{(1 - k\frac{m}{R})^2}. \tag{C.3}$$

As for the spatial derivatives we have

$$\boldsymbol{R}_x = \boldsymbol{e}_1 - \alpha\dot{\boldsymbol{\psi}}(\dot{\boldsymbol{\psi}},\boldsymbol{e}_1), \qquad \boldsymbol{R}_{xx} = 0, \tag{C.4}$$

where e_1 is a unit vector on the x axis.

$$f_x = -\frac{m}{R^3}\cdot\frac{(\boldsymbol{R},\boldsymbol{R}_x)}{1 - k\frac{m}{R}} \tag{C.5}$$

$$f_{xx} = 3\frac{m}{R^5} \cdot \frac{(\boldsymbol{R}, \boldsymbol{R}_x)^2}{1-k\frac{m}{R}} - \frac{m}{R^3} \cdot \frac{(\boldsymbol{R}_x, \boldsymbol{R}_x)}{1-k\frac{m}{R}} + \frac{km^2}{R^6} \cdot \frac{(\boldsymbol{R}, \boldsymbol{R}_x)^2}{(1-k\frac{m}{R})^2}. \tag{C.6}$$

Thus

$$\begin{aligned} \triangle f &= 3\frac{m}{R^5} \cdot \frac{(\boldsymbol{R}, \boldsymbol{R}_x)^2 + (\boldsymbol{R}, \boldsymbol{R}_y)^2 + (\boldsymbol{R}, \boldsymbol{R}_z)^2}{1-k\frac{m}{R}} - \\ & \frac{m}{R^3} \cdot \frac{(\boldsymbol{R}_x, \boldsymbol{R}_x) + (\boldsymbol{R}_y, \boldsymbol{R}_y) + (\boldsymbol{R}_z, \boldsymbol{R}_z)}{1-k\frac{m}{R}} + \\ & \frac{km^2}{R^6} \cdot \frac{(\boldsymbol{R}, \boldsymbol{R}_x)^2 + (\boldsymbol{R}, \boldsymbol{R}_y)^2 + (\boldsymbol{R}, \boldsymbol{R}_z)^2}{(1-k\frac{m}{R})^2} \end{aligned} \tag{C.7}$$

Substitute the value of $\boldsymbol{R}_x$ to get

$$\begin{aligned} \triangle f &= 3\frac{m}{R^5} \cdot \frac{R^2 - 2\alpha(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2 + \alpha^2|\dot{\boldsymbol{\psi}}|^2(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2}{1-k\frac{m}{R}} - \frac{m}{R^5} \cdot \frac{3 - 2\alpha|\dot{\boldsymbol{\psi}}|^2 + \alpha^2|\dot{\boldsymbol{\psi}}|^4}{1-k\frac{m}{R}} + \\ & \frac{km^2}{R^6} \cdot \frac{R^2 - 2\alpha(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2 + \alpha^2|\dot{\boldsymbol{\psi}}|^2(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2}{(1-k\frac{m}{R})^2} \end{aligned} \tag{C.8}$$

Thus the second order l.h.s. of the equation is

$$\begin{aligned} \Box f &= \frac{m\beta}{R^5} \cdot \frac{3\beta(\boldsymbol{R}, \dot{\boldsymbol{\psi}})^2 + R^2(\boldsymbol{R}, \ddot{\boldsymbol{\psi}}) - \beta R^2|\dot{\boldsymbol{\psi}}|^2}{1-k\frac{m}{R}} + \frac{m^2\beta^2 k}{R^6} \cdot \frac{(\boldsymbol{R}, \dot{\boldsymbol{\psi}})^2}{(1-k\frac{m}{R})^2} \\ & -3\frac{m}{R^5} \cdot \frac{R^2 - 2\alpha(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2 + \alpha^2|\dot{\boldsymbol{\psi}}|^2(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2}{1-k\frac{m}{R}} + \frac{m}{R^3} \cdot \frac{3 - 2\alpha|\dot{\boldsymbol{\psi}}|^2 + \alpha^2|\dot{\boldsymbol{\psi}}|^4}{1-k\frac{m}{R}} - \\ & \frac{km^2}{R^6} \cdot \frac{R^2 - 2\alpha(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2 + \alpha^2|\dot{\boldsymbol{\psi}}|^2(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2}{(1-k\frac{m}{R})^2} \\ &= \frac{m\beta}{R^3} \cdot \frac{(\boldsymbol{R}, \ddot{\boldsymbol{\psi}})}{1-k\frac{m}{R}} + \frac{m}{R^5}(\beta^2 + 2\alpha - \alpha^2|\dot{\boldsymbol{\psi}}|^2) \cdot \frac{(3(\boldsymbol{R}, \dot{\boldsymbol{\psi}})^2 - R^2|\dot{\boldsymbol{\psi}}|^2)}{1-k\frac{m}{R}} + \\ & \frac{m^2 k}{R^6} \cdot \frac{-R^2 + (\beta^2 + 2\alpha - \alpha^2|\dot{\boldsymbol{\psi}}|^2)(\boldsymbol{R}, \dot{\boldsymbol{\psi}})^2}{(1-k\frac{m}{R})^2} \end{aligned} \tag{C.9}$$

Using the relation $\beta^2 = \alpha^2|\dot{\boldsymbol{\psi}}|^2 - 2\alpha$ we obtain

$$\Box f = \frac{m\beta}{R^3} \cdot \frac{(\boldsymbol{R}, \ddot{\boldsymbol{\psi}})}{1-k\frac{m}{R}} - \frac{m^2 k}{R^4} \frac{1}{(1-k\frac{m}{R})^2} \tag{C.10}$$

As for the quadratic r.h.s.

$$\begin{aligned} \eta^{ab} f_{,a} f_{,b} &= \frac{m^2\beta^2}{R^6} \cdot \frac{(\boldsymbol{R}, \dot{\boldsymbol{\psi}})^2}{(1-k\frac{m}{R})^2} - \frac{m^2}{R^6} \cdot \frac{(\boldsymbol{R}, \boldsymbol{R}_x)^2 + (\boldsymbol{R}, \boldsymbol{R}_y)^2 + (\boldsymbol{R}, \boldsymbol{R}_z)^2}{(1-k\frac{m}{R})^2} \\ &= \frac{m^2\beta^2}{R^6} \cdot \frac{(\boldsymbol{R}, \dot{\boldsymbol{\psi}})^2}{(1-k\frac{m}{R})^2} - \frac{m^2}{R^6} \cdot \frac{R^2 - 2\alpha(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2 + \alpha^2|\dot{\boldsymbol{\psi}}|^2(\dot{\boldsymbol{\psi}}, \boldsymbol{R})^2}{(1-k\frac{m}{R})^2} \\ &= -\frac{m^2}{R^4} \frac{1}{(1-k\frac{m}{R})^2} + \frac{m^2}{R^6}(\beta^2 + 2\alpha - \alpha^2|\dot{\boldsymbol{\psi}}|^2) \frac{(\boldsymbol{R}, \dot{\boldsymbol{\psi}})^2}{(1-k\frac{m}{R})^2} \end{aligned} \tag{C.11}$$

Thus, to the first order,

$$\eta^{ab} f_{,a} f_{,b} = -\frac{m^2}{R^4} \frac{1}{(1-k\frac{m}{R})^2} \tag{C.12}$$

D The quadratic part

Calculate the nonlinear part

$$f_t = -\sum_{i=1}^{n} \Big(\frac{({}^{(i)}\boldsymbol{R}_t, {}^{(i)}\boldsymbol{R})}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} \cdot \frac{{}^{(i)}m}{{}^{(i)}R^3} \Big) = \sum_{i=1}^{n} \Big(\frac{({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R})}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} \cdot \frac{{}^{(i)}m{}^{(i)}\beta}{{}^{(i)}R^3} \Big). \tag{D.1}$$

Thus

$$(f_t)^2 = \sum_{i,j=1}^{n} \frac{\frac{{}^{(i)}m{}^{(i)}\beta}{{}^{(i)}R^3}}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} \cdot \frac{\frac{{}^{(j)}m{}^{(j)}\beta}{{}^{(j)}R^3}}{1 - k\frac{{}^{(j)}m}{{}^{(j)}R}} ({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R})({}^{(j)}\dot{\boldsymbol{\psi}}, {}^{(j)}\boldsymbol{R}). \tag{D.2}$$

As for the spatial derivatives

$$f_x = -\sum_{i=1}^{n} \frac{({}^{(i)}\boldsymbol{R}_x, {}^{(i)}\boldsymbol{R})}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} \cdot \frac{{}^{(i)}m}{{}^{(i)}R^3} \tag{D.3}$$

$$\begin{aligned} (\nabla f, \nabla f) \quad &= \quad \sum_{i,j=1}^{n} \frac{\frac{{}^{(i)}m}{{}^{(i)}R^3}}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} \cdot \frac{\frac{{}^{(j)}m}{{}^{(j)}R^3}}{1 - k\frac{{}^{(j)}m}{{}^{(j)}R}} \Big(({}^{(i)}\boldsymbol{R}_x, {}^{(i)}\boldsymbol{R})({}^{(j)}\boldsymbol{R}_x, {}^{(j)}\boldsymbol{R}) + \\ &\quad ({}^{(i)}\boldsymbol{R}_y, {}^{(i)}\boldsymbol{R})({}^{(j)}\boldsymbol{R}_y, {}^{(j)}\boldsymbol{R}) + ({}^{(i)}\boldsymbol{R}_z, {}^{(i)}\boldsymbol{R})({}^{(j)}\boldsymbol{R}_z, {}^{(j)}\boldsymbol{R}) \Big) \end{aligned} \tag{D.4}$$

Using the relation

$${}^{(i)}\boldsymbol{R}_x = \boldsymbol{e}_1 - {}^{(i)}\alpha{}^{(i)}\dot{\boldsymbol{\psi}}({}^{(i)}\dot{\boldsymbol{\psi}}, \boldsymbol{e}_1) \tag{D.5}$$

and writing

$$({}^{(i)}\boldsymbol{R}_x, {}^{(i)}\boldsymbol{R}) = (\boldsymbol{e}_1, {}^{(i)}\boldsymbol{R}) - {}^{(i)}\alpha({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R})({}^{(i)}\dot{\boldsymbol{\psi}}, \boldsymbol{e}_1) \tag{D.6}$$

we obtain

$$\begin{aligned} \Big(({}^{(i)}\boldsymbol{R}_x, {}^{(i)}\boldsymbol{R})({}^{(j)}\boldsymbol{R}_x, {}^{(j)}\boldsymbol{R}) \Big) &= \Big((\boldsymbol{e}_1, {}^{(i)}\boldsymbol{R})(\boldsymbol{e}_1, {}^{(j)}\boldsymbol{R}) \Big) - \\ &\quad {}^{(j)}\alpha \Big((\boldsymbol{e}_1, {}^{(i)}\boldsymbol{R})({}^{(j)}\dot{\boldsymbol{\psi}}, {}^{(j)}\boldsymbol{R})({}^{(j)}\dot{\boldsymbol{\psi}}, \boldsymbol{e}_1) \Big) - \\ &\quad {}^{(i)}\alpha \Big((\boldsymbol{e}_1, {}^{(j)}\boldsymbol{R})({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R})({}^{(i)}\dot{\boldsymbol{\psi}}, \boldsymbol{e}_1) \Big) + \\ &\quad {}^{(i)}\alpha{}^{(j)}\alpha({}^{(i)} \Big(\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R})({}^{(i)}\dot{\boldsymbol{\psi}}, \boldsymbol{e}_1)({}^{(j)}\dot{\boldsymbol{\psi}}, {}^{(j)}\boldsymbol{R})({}^{(j)}\dot{\boldsymbol{\psi}}, \boldsymbol{e}_1) \Big). \end{aligned} \tag{D.7}$$

Thus the brackets in (D.4) are

$$\begin{aligned} \Big(\quad \Big) \quad &= \quad ({}^{(i)}\boldsymbol{R}, {}^{(j)}\boldsymbol{R}) - {}^{(j)}\alpha \Big(({}^{(j)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R})({}^{(j)}\dot{\boldsymbol{\psi}}, {}^{(j)}\boldsymbol{R}) \Big) - \\ &\quad {}^{(i)}\alpha \Big(({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(j)}\boldsymbol{R})({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R}) \Big) + \\ &\quad {}^{(i)}\alpha{}^{(j)}\alpha \Big(({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R})({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(j)}\dot{\boldsymbol{\psi}})({}^{(j)}\dot{\boldsymbol{\psi}}, {}^{(j)}\boldsymbol{R}) \Big). \end{aligned} \tag{D.8}$$

Consequently, the r.h.s. of the field equation is

$$\begin{aligned} k\eta^{ab} f_{,a} f_{,b} \quad &= \quad -\sum_{i,j=1}^{n} \frac{\frac{{}^{(i)}m}{{}^{(i)}R^3}}{1 - k\frac{{}^{(i)}m}{{}^{(i)}R}} \cdot \frac{\frac{{}^{(j)}m}{{}^{(j)}R^3}}{1 - k\frac{{}^{(j)}m}{{}^{(j)}R}} \Big[({}^{(i)}\boldsymbol{R}, {}^{(j)}\boldsymbol{R}) + \\ &\quad ({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(j)}\boldsymbol{R})({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(i)}\boldsymbol{R}) \Big({}^{(i)}\beta{}^{(j)}\beta + {}^{(i)}\alpha + {}^{(j)}\alpha - {}^{(i)}\alpha{}^{(j)}\alpha({}^{(i)}\dot{\boldsymbol{\psi}}, {}^{(j)}\dot{\boldsymbol{\psi}}) \Big) \Big] \end{aligned}$$

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